

ON THE HOFER GEOMETRY INJECTIVITY RADIUS CONJECTURE

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ABSTRACT. We verify here some variants of topological and dynamical flavor of the injectivity radius conjecture in Hofer geometry, Lalonde-Savelyev [4] in the case of $\text{Ham}(S^2, \omega)$ and $\text{Ham}(\Sigma, \omega)$, for Σ a closed positive genus surface. In particular we show that any loop in $\text{Ham}(S^2, \omega)$, respectively $\text{Ham}(\Sigma, \omega)$ with L^+ Hofer length less than $\text{area}(S^2)/2$, respectively any L^+ length is contractible through (L^+) Hofer shorter loops, in the C^∞ topology. We also prove some stronger variants of this statement on the loop space level. One dynamical type corollary is that there are no smooth, positive Morse index (Ustilovsky) geodesics, in $\text{Ham}(S^2, \omega)$, respectively in $\text{Ham}(\Sigma, \omega)$ with L^+ Hofer length less than $\text{area}(S^2)/2$, respectively any length. The above condition on the geodesics can be expanded as an explicit and elementary dynamical condition on the associated Hamiltonian flow. We also give some speculations on connections of this later result with curvature properties of the Hamiltonian diffeomorphism group of surfaces.

1. INTRODUCTION

One of the most fundamental objects associated to a Finsler manifold is its injectivity radius function. The group $\text{Ham}(M, \omega)$, with its bi-invariant Finsler Hofer metric, has no well defined exponential map, so that if we ask about its injectivity radius we need to decide how to interpret this. One simple way to interpret is to ask for the size of the largest metric epsilon-ball which is contractible, or has null-homotopic inclusion map into the total space $\text{Ham}(M, \omega)$.

In this formulation we studied the question of injectivity radius for $\text{Ham}(M, \omega)$ in Lalonde-Savelyev [4], and formulated there the conjecture that the injectivity and (or) weak injectivity radius of $\text{Ham}(M, \omega)$ is positive.

In this note we shall verify some variants of this conjecture in the case of surfaces. The main new ingredient is a kind of “curvature” ¹ flow for connection type symplectic forms on surface Hamiltonian fibrations over S^2 , which exists in the presence of certain foliations by holomorphic curves, using which we also obtain a curve shortening algorithm. It is interesting to speculate whether this is at all related to symplectic Ricci type flows, but to emphasize, our “flow” is of very elementary nature, at least assuming the state-of-the-art in Gromov-Witten theory.

Using these developments we obtain some dynamical applications, which are formulated in terms of certain non-existence results of Ustilovsky geodesics. These may also be interpreted as injectivity radius statements but from a Finsler geometric rather than purely metric point of view. This also uses the author’s (virtual) Morse

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¹This is a slight misnomer as curvature may not be monotonically decreasing under flow. But probably the best way to think about it.

theory for the Hofer length functional. We follow this with some speculations on the “curvature” of the Hamiltonian group of surfaces.

1.1. Statements. Given a smooth function $H : M^{2n} \times (S^1 = \mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{R}$, (time is in S^1 for convenience and later use) there is an associated time-dependent Hamiltonian vector field X_t , $0 \leq t \leq 1$, defined by

$$(1.1) \quad \omega(X_t, \cdot) = -dH_t(\cdot).$$

The vector field X_t generates a path $\gamma : [0, 1] \rightarrow \text{Diff}(M)$, starting at id. Given such a path γ , its end point $\gamma(1)$ is called a Hamiltonian symplectomorphism. The space of Hamiltonian symplectomorphisms forms a group, denoted by $\text{Ham}(M, \omega)$. In particular the path γ above lies in $\text{Ham}(M, \omega)$. Given a general smooth path γ , the *positive Hofer length*, $L^+(\gamma)$ is defined by

$$L^+(\gamma) := \int_0^1 \max_M H_t^\gamma dt,$$

where H^γ is a generating function for the path $t \mapsto \gamma(0)^{-1}\gamma(t)$, $0 \leq t \leq 1$, normalized to have zero mean at each moment.

Part of the reason for interest in the above functional as well as in the original Hofer length functional, which is obtained by integration of the L^∞ norm of the generating function, is due to existence of some deep connections of this with both the theory of dynamical systems and with Floer-Gromov-Witten theory.

Notation 1.1. *Let*

$$\Omega^c \text{Ham}(M, \omega) := \{\gamma \in \Omega \text{Ham}(M, \omega) \mid L^+(\gamma) < c\}.$$

This is taken in the topology induced by the C^∞ topology.

Definition 1.2. *Let*

$$\text{injrad}_{\Omega,+}(M, \omega) := \sup\{c \mid \Omega^{c'} \text{Ham}(M, \omega) \text{ is (intrinsically) contractible for } c' \leq c\}.$$

Theorem 1.3. *Set $\hbar = \text{area}(S^2, \omega)$, and let Σ denote a positive genus surface, then we have*

$$(1.2) \quad \text{injrad}_{\Omega,+}(S^2) = \hbar/2,$$

$$(1.3) \quad \text{injrad}_{\Omega,+}(\Sigma) = \infty.$$

Corollary 1.4. *In particular a smooth loop $\gamma \in \Omega^{\hbar/2} \text{Ham}(S^2, \omega)$, respectively in $\Omega \text{Ham}(\Sigma, \omega)$ with any L^+ Hofer length, is contractible through Hofer shorter loops.*

The above theorem is implied by the following theorem once we observe that any loop with positive Hofer length less than $\hbar/2$ is contractible, which is a classical fact proved for example via Seidel morphism [15].

Theorem 1.5. *Denote by*

$$\Omega^{c, \text{cont}} \text{Ham}(S^2, \omega) \subset \Omega^c \text{Ham}(S^2, \omega)$$

the subspace of contractible loops, then $\Omega^{c, \text{cont}} \text{Ham}(S^2, \omega)$ is contractible for any $c \leq \hbar$.

We can readily see that this bound \hbar is optimal, as we have a natural representative f_{\min} of a generator of

$$\pi_2(\Omega \text{Ham}(S^2, \omega)) = \pi_3(\text{Ham}(S^2, \omega)) \simeq \mathbb{Z}$$

all of whose loops are contractible and have L^+ Hofer length at most \hbar . Let us make this more explicit. Take the natural Lie group homomorphism $\tilde{f} : S^3 \rightarrow \text{Ham}(S^2, \omega)$, representing the generator of $\pi_3(\text{Ham}(S^2, \omega))$. Deloop this to a map $f_{\min} : S^2 \rightarrow \Omega \text{Ham}(S^2, \omega)$, by taking the natural S^2 family of based at id loops on S^3 , so that the longest loop in this family corresponds to a simple great geodesic γ , for the round metric. In other words this is the family forming the unstable manifold for γ , which is an index 2 geodesic. Then $f_{\min}(\gamma)$ is the longest loop in the image of S^2 , and its positive Hofer length is $2\hbar/2 = \hbar$. Consequently we get:

Corollary 1.6. *The map f_{\min} cannot be homotoped into $\Omega^{\hbar} \text{Ham}(S^2, \omega)$.*

We also proved this using a more general theory of quantum characteristic classes in [11].

1.1.1. *Dynamical consequences.* It is shown by Ustilovsky [16] that γ is a smooth critical point of L^+ Hofer length functional on the smooth path space from id to ϕ , if there is a point $x_{\max} \in M$ maximizing the generating function H_t^γ at each moment t , and such that H_t^γ is Morse at x_{\max} , at each moment t .

Definition 1.7. *We call such a γ an L^+ Ustilovsky geodesic.*

As an Ustilovsky geodesic is a smooth critical point it makes sense to ask for its Morse index. The index was shown by the author in [14] to be finite, in fact for closed Ustilovsky geodesics in the group of Hamiltonian diffeomorphisms of a surface we have:

Theorem 1.8. *Let $\gamma \in \Omega \text{Ham}(M, \omega)$ be a smooth closed L^+ Ustilovsky geodesic, where M is a surface. Let γ_* denote the linearization of γ at x_{\max} , which is a loop of linear symplectomorphisms of $T_x M$. Then the Morse index of γ with respect to L^+ (and conventions (2.1)) is*

$$-\text{Maslov}(\gamma_*) - 2,$$

if $\text{Maslov}(\gamma_) \leq -2$, otherwise the Morse index is 0.*

Here the Maslov number is normalized so that for the clockwise single rotation of \mathbb{R}^2 the Maslov number is -2. The above theorem can be readily deduced from [14].

Definition 1.9. *We will say that a smooth closed Ustilovsky L^+ geodesic*

$$\gamma : S^1 \rightarrow \text{Ham}(M, \omega)$$

*is **quasi-integrable**, if there is a Darboux chart $\phi : (U \subset \mathbb{C}^n) \rightarrow M$, $\phi(0) = x_{\max}$ at the extremizer x_{\max} , in which the generating function H^γ coincides with its Hessian quadratic form in some neighborhood of 0, and this quadratic form is the real part of a complex quadratic form, or in other words if the Hamiltonian flow for this Hessian quadratic form is unitary.*

Thus this is a kind of Morse-Darboux integrability condition at x_{\max} , and is automatic when γ is a circle action near x_{\max} .

Theorem 1.10. *There are no non-constant, smooth, closed, positive Morse index L^+ Ustilovsky geodesics, in $\text{Ham}(S^2, \omega)$, respectively in $\text{Ham}(\Sigma, \omega)$, with L^+ Hofer length less than $\hbar/2$, respectively any length.*

The same statement holds for index 0 geodesics if they are quasi-integrable Ustilovsky.

This will be proved in Section 3. The positive Morse index condition on Ustilovsky geodesic is in a sense an elementary dynamical condition. So the previous theorem is really an elementary statement in Hamiltonian dynamics. It would be interesting to understand how it extends to more general geodesics, or even what the appropriate generalization of Ustilovsky geodesics is. For example McDuff and Lalonde consider certain generalizations of Ustilovsky geodesics in [3], but these seem to be ill adapted to the variational style arguments that we make. We have also avoided talking about non closed geodesics, for while Savelyev [13] is well adapted to the case of paths, our arguments here are not, and do not immediately generalize.

Remark 1.11. *Very heuristically the statement for $\text{Ham}(\Sigma, \omega)$ suggests that the “curvature” of $\text{Ham}(\Sigma, \omega)$ with respect to the positive Hofer length functional is non-positive. In a finite dimensional setting this could be justified via Gauss-Bonnet theorem. See also Milnor’s [8] for background on connections of curvature and topology of loop spaces. Similarly we may interpret the statement for $\text{Ham}(S^2, \omega)$ as a certain upper bound on the positivity of “curvature”. However what is “curvature”? We can try to think of this in terms of coarse geometry, the dream is something like:*

Conjecture 1.12. *(Preliminary) The space $\text{Ham}(T^2, \omega)$ is a rough $\text{CAT}(0)$ space, and $\text{Ham}(\Sigma_g, \omega)$ is a rough $\text{CAT}(k)$ space for $k \leq 0$, for $g > 1$, while $\text{Ham}(S^2, \omega)$ is a rough $\text{CAT}(k)$ space for some $k > 0$, see [1], with respect to the positive Hofer length functional.*

2. PROOF OF THEOREM 1.5

We first verify the case of $M = S^2$. The following argument works the same way for any $0 < c \leq \hbar$, we shall just do it with $c = \hbar$. To recall, our conventions for the Hamiltonian flow and compatible almost complex structures are:

$$(2.1) \quad \omega(X_H, \cdot) = -dH(\cdot)$$

$$(2.2) \quad \omega(v, Jv) > 0, \text{ for } v \neq 0.$$

For every $l \in \Omega^{\hbar, \text{cont}} \text{Ham}(S^2)$ we get a Hamiltonian S^2 fibration X_l over \mathbb{CP}^1 , by using l as a clutching map:

$$X_l = S^2 \times D_-^2 \sqcup S^2 \times D_+^2 / \sim.$$

Recall that a *coupling form*, see [2] for a Hamiltonian fibration $M \rightarrow P \xrightarrow{\pi} X$ is a closed 2-form $\tilde{\alpha}$ on the total space, which restricts to the symplectic form on the fibers, modeled by (M, ω) , and which satisfies:

$$\pi_* \tilde{\alpha} = 0,$$

where π_* is the integration over the fiber map. By [2] such forms $\tilde{\alpha}$ always exist for a Hamiltonian fibration, and are uniquely determined by the associated Hamiltonian connections, defined by declaring the horizontal subspaces to be the $\tilde{\alpha}$ -orthogonal

subspaces to the vertical tangent spaces. We have the coupling form $\tilde{\Omega}_{l,-}$ on $S^2 \times D_-^2$ defined by

$$\tilde{\Omega}_{l,-} = \omega - d(\eta(r) \cdot H^l d\theta),$$

where $0 \leq r \leq 1, 0 \leq \theta \leq 1$, (which are our modified polar coordinates) H^l is the normalized generating function for l , and $\eta : [0, 1] \rightarrow [0, 1]$ is a smooth function satisfying:

$$0 \leq \eta'(r),$$

and

$$(2.3) \quad \eta(r) = \begin{cases} r^2 & \text{if } 0 \leq r \leq 1 - 2\kappa \\ 1 & \text{if } 1 - \kappa \leq r \leq 1, \end{cases}$$

for a small $\kappa > 0$. Under the gluing relation \sim , $\tilde{\Omega}_{l,-}$ corresponds to the form ω near the boundary of $S^2 \times D_+^2$, so we may extend trivially over D_+^2 to get a coupling form $\tilde{\Omega}_l$ on X_l . The coupling form $\tilde{\Omega}_l$ determines a Hamiltonian connection \mathcal{A}_l as described above. This in turn determines an almost complex structure J_l , by first fixing a family $\{j_l(z)\}$, $z \in \mathbb{CP}^1$ smooth in z : $j_l(z)$ is an almost complex structure on the fiber $\pi_l^{-1}(z)$, where

$$\pi_l : X_l \rightarrow \mathbb{CP}^1$$

is the projection, with each $j_l(z)$ compatible with the symplectic form on the fiber, and then defining J_l to coincide with $\{j_l(z)\}$ on the vertical tangent bundle, to preserve the horizontal distribution of \mathcal{A}_l and to have a holomorphic projection map to (\mathbb{CP}^1, j) , where j is the almost complex structure which preserves the standard orientation on \mathbb{CP}^1 . As each l is contractible and so X_l trivializable, we may consider the moduli spaces

$$\overline{\mathcal{M}}(J_l),$$

consisting of stable J_l holomorphic sections σ of X_l , in the class of the constant section $[const]$, and in particular satisfying:

$$\langle [\tilde{\Omega}_l], [\sigma] \rangle = 0.$$

By a *stable holomorphic section* we mean a stable J_l -holomorphic map σ into X_l , in the classical sense, [6] with domain an unmarked nodal Riemann sphere, one of those components is called *principal*. The restriction σ_{princ} of σ , to the principal component is a J_l holomorphic section, i.e. we have a commutative diagram:

$$\begin{array}{ccc} & X_l & \\ \sigma_{princ} \nearrow & \downarrow \pi_l & \\ \mathbb{CP}^1 & \xrightarrow{\text{id}} & \mathbb{CP}^1. \end{array}$$

All the other components of σ are vertical, that is they are J_l -holomorphic maps into the fibers of X_l .

Lemma 2.1. *The moduli space $\overline{\mathcal{M}}(J_l)$ has no nodal curves as elements.*

Proof. Suppose otherwise, then there is a stable J_l -holomorphic section σ of X_l , with total homology class $[const]$, and consequently having a principal component σ_{princ} which is a smooth J_l -holomorphic section of X_l with:

$$\langle [\tilde{\Omega}_l], [\sigma_{princ}] \rangle \leq -\hbar$$

as \hbar is the minimal energy of a non-constant holomorphic sphere in S^2 , i.e. $\text{area}(S^2, \omega)$. However in this case the classical energy inequality for holomorphic curves gives:

$$\hbar \leq \text{area}^+(\tilde{\Omega}_l),$$

where area^+ is the functional on the space of coupling forms

$$\text{area}^+(\tilde{\Omega}) = \inf_{\alpha} \left\{ \int_{\mathbb{CP}^1} \alpha \mid \tilde{\Omega} + \pi^*(\alpha) \text{ is symplectic} \right\},$$

for α a 2-form on \mathbb{CP}^1 , with positive integral. On the other hand by direct calculation we have

$$\text{area}^+(\tilde{\Omega}_l) = L^+(l) < \hbar.$$

This gives a contradiction. \square

By automatic transversality see [6, Appendix C] $\overline{\mathcal{M}}(J_l)$ is regular i.e. the associated real linear Cauchy-Riemann operator is transverse for all $\sigma \in \overline{\mathcal{M}}(J_l)$. As these are embedded we may use positivity of intersections, see [6, Section 2.6], to infer that $\overline{\mathcal{M}}(J_l)$ determines a smooth foliation of X_l by $[const]$ class holomorphic sections. In particular

$$ev : \overline{\mathcal{M}}(J_l) \rightarrow S^2,$$

obtained by evaluating a section at $0 \in \mathbb{CP}^1$, is a diffeomorphism, and determines a canonical smooth trivialization of X_l . Let Θ_l denote the corresponding horizontal distribution.

Consequently for an appropriately smooth family:

$$f : S^k \rightarrow \Omega^{\hbar, cont} \text{Ham}(S^2, \omega),$$

we get natural (up to choices of almost complex structures) smooth trivialization of the bundle

$$P_f \rightarrow S^k,$$

with fiber over $s \in S^k$ being $X_s \equiv X_{f(s)}$. This is a trivialization of a bundle with structure group $\Omega^2 \text{Diff}(S^2)$. What is important for what follows is that this structure group is a subgroup of the group of smooth bundle maps of $S^2 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$.

Let

$$tr : P_f \rightarrow (S^2 \times \mathbb{CP}^1) \times S^k,$$

denote this trivialization. Set $\{\mathcal{F}_s = (tr_s)^*\omega\}$, where tr_s denotes the restriction of tr to the fiber X_s . By the above tr_s is a smooth bundle map and hence each \mathcal{F}_s is a *connection type* closed form, which just means the restriction to each fiber of $\pi_s : X_s \rightarrow \mathbb{CP}^1$, is symplectic. This in turn means that the smooth bundle connection determined by declaring horizontal spaces to be \mathcal{F}_s -orthogonal spaces to the vertical tangent spaces, is Hamiltonian (with respect to the family of fiber symplectic forms determined by the restrictions.)

Also by construction each \mathcal{F}_s vanishes on the horizontal distribution $\Theta_{f(s)}$, and so

$$\tau_s = \mathcal{F}_s + \rho \pi_s^* \tau,$$

is a symplectic form on X_s for any small $\rho > 0$, and where τ a fixed, area one symplectic form on \mathbb{CP}^1 . Likewise

$$\alpha_s = \tilde{\Omega}_{f(s)} + \text{area}^+(\tilde{\Omega}_{f(s)})\pi_s^*\tau + \rho\pi_s^*\tau,$$

is symplectic, and α_s is positive on the horizontal distribution $\Theta_{f(s)}$.

Let $\alpha_{s,t}$, $t \in [0, 1]$ denote the convex linear combination

$$\alpha_{s,t} = (t)\tau_s + (1-t)\alpha_s.$$

Note that $\alpha_{s,t}$ is nondegenerate and hence symplectic for every s, t , as this is a convex linear combination of symplectic forms tamed by the same almost complex structure $J(\mathcal{A}_s)$, where \mathcal{A}_s is the connection determined by $\tilde{\Omega}_{f(s)}$ as before, and so $\alpha_{s,t}$ is also tamed by this complex structure. Moreover for a fixed fiber S_z^2 , $\alpha_{s,t}$ is a convex linear combination of cohomologous symplectic forms $\mathcal{F}_s|_{S_z^2}$ and $\tilde{\Omega}_{f(s)}|_{S_z^2}$. By construction $(S_0^2, \tilde{\Omega}_{f(s)}|_{S_0^2})$ is naturally symplectomorphic to (S^2, ω) , consequently applying Moser's argument we get that $\alpha_{s,t}|_{S_0^2}$ is naturally symplectomorphic to (S^2, ω) , for each t . We shall use this further on in the “radial trivialization” construction.

Notation 2.2. Denote by $\tilde{\alpha}_{s,t}$ the coupling form determined by $\alpha_{s,t}$.

Lemma 2.3. $\text{area}^+(\tilde{\alpha}_{s,t})$ is bounded from above by a function $b(s, t)$ non-increasing with t , with $b(s, 1) = \rho$, and with $b(s, 0) = L^+(f(s)) + \rho$.

Proof. Define

$$(2.4) \quad \text{area}(\alpha) := \text{Vol}(X_s, \alpha) / \text{Vol}(S^2, \omega).$$

Note that

$$\text{area}^+(\tilde{\alpha}_{s,t}) \leq \text{area}(\alpha_{s,t}),$$

which follows by the fact that

$$\alpha_{s,t} = \tilde{\alpha}_{s,t} + \pi_s^* \tau',$$

for an area form τ' by the construction. Set $b(s, t) = \text{area}(\alpha_{s,t})$, then by direct calculation

$$b(s, 0) = L^+(f(s)) + \rho,$$

and

$$b(s, 1) = \rho.$$

□

Each $\tilde{\alpha}_{s,t}$ determines a loop $f_{s,t}$ in $\text{Ham}(S^2, \omega)$, first by using the natural identification of $(S_0^2, \tilde{\alpha}_{s,t}|_{S_0^2})$ with (S^2, ω) as described above, then identifying the fiber over $0 \in \mathbb{CP}^1$ with the fiber over $\infty \in \mathbb{CP}^1$ by $\tilde{\alpha}_{s,t}$ -parallel transport map over the $\theta = 0$ ray from 0 to ∞ ($0 = 0 \in D_-^2$, $\infty = 0 \in D_+^2$ in our coordinates from before), and then for every other θ ray from 0 to ∞ getting an element $f_{s,t}(\theta) \in \text{Ham}(S^2, \omega)$ by $\tilde{\alpha}_{s,t}$ parallel transport over this ray. Clearly $f_{s,0} = f(s)$, and $f_{s,1} = \text{id}$ and for each t , $f(s, t)$ is a loop of Hamiltonian symplectomorphisms of (S^2, ω) (by the identification above.) On the other hand by the proof of [10, Lemma 3.3A] (cf. proof of [5, Lemma 2.2]) we have:

$$(2.5) \quad L^+(f_{s,t}) \leq \text{area}^+(\tilde{\alpha}_{s,t}).$$

Consequently, by the Lemma 2.3 we get that $L^+(f_{s,t})$ as a function of t is bounded from above by the non-increasing continuous function $b(s, t)$ with maximum $L^+(f(s)) + \rho$, and with $b(s, 1) = \rho$. Taking ρ small enough so that $L^+(f_s) + \rho < \hbar$ for each s , we obtain a null-homotopy of f in $\Omega^{\hbar, \text{cont}} \text{Ham}(S^2, \omega)$.

So we get that $\Omega^{h,cont} \text{Ham}(S^2, \omega)$ is weakly contractible. Next note that $\Omega^h \text{Ham}(S^2, \omega)$ is an open subset of $\Omega \text{Ham}(S^2, \omega)$, which by Milnor [9], has the homotopy type of a CW complex, as it is the loop space of a Frechet manifold, hence of an absolute neighborhood retract, and so $\Omega^h \text{Ham}(S^2, \omega)$ has the homotopy type of a CW complex. So it follows that $\Omega^{h,cont} \text{Ham}(S^2, \omega)$ is contractible.

For a Σ as in the statement of the theorem, we may proceed with exactly the same argument upon noting that now there is no bubbling at all, so that we don't need to restrict to short loops. \square

3. A VIRTUALLY PERFECT MORSE THEORY FOR THE HOFER LENGTH FUNCTIONAL AND POSITIVE MORSE INDEX GEODESICS IN $\text{Ham}(S^2, \omega), \text{Ham}(\Sigma, \omega)$

Let us review our notion of a local unstable manifold in the setting of loops.

Definition 3.1. Let $\gamma \in \Omega \text{Ham}(M, \omega)$ be an index k , L^+ Ustilovsky geodesic and let B^k denote the standard k -ball in \mathbb{R}^k , centered at the origin 0. Let $\Omega \text{Ham}(M, \omega)_E$ denote the E sub-level set of the loop space, with respect to L^+ , with

$$0 < E < L^+(\gamma),$$

where by E sub-level set we mean the set of elements $\gamma \in \Omega \text{Ham}(M, \omega)_E$ satisfying $L^+(\gamma) \leq E$. A **local unstable manifold** at γ is a pair (f_γ, E) , with

$$f_\gamma : B^k \rightarrow \Omega \text{Ham}(M, \omega),$$

s.t.

$$f_\gamma(0) = \gamma,$$

$f_\gamma^* L^+$ is a function Morse at the unique maximum $0 \in B^k$, and s.t.

$$f_\gamma(\partial B^k) \subset \Omega \text{Ham}(M, \omega)_E.$$

It is explained in [13] that local unstable manifolds always exist.

An interesting consequence of the “index theorem” is that for closed Ustilovsky geodesics the Morse index must be even, and this gives a bit of intuition into the following theorem which can be interpreted as saying that the Hofer length functional on the loop space of the Hamiltonian group of a surface is a kind of perfect Morse-Bott function. We say surface for while the index stays even in general the following theorem does not hold without an additional technical assumption on the geodesic. (We need the kernel of certain Cauchy-Riemann operator associated to the geodesic to be trivial.)

The following is a version of the author's [13, Theorem 1.9], in the case of surfaces, where it becomes a stronger result, due to the additional input of automatic transversality.

Theorem 3.2. Let M be any Riemann surface. Let $\gamma \in \Omega \text{Ham}(M, \omega)$ be a closed smooth positive Morse index k , L^+ Ustilovsky geodesic. If (f_γ, E) is a local unstable manifold for γ then

$$0 \neq [f_\gamma] \in \pi_k(\Omega \text{Ham}(M, \omega), \Omega \text{Ham}(M, \omega)_E).$$

If the Morse index of γ is 0 and γ is quasi-integrable, then γ is L^+ length minimizing. This second part of the theorem holds for a general symplectic manifold (assuming virtual techniques.)

The index 0 case of the above is just a very minor variation of a foundational result in McDuff-Slimowitz [7], and must be well known to experts. A relatively elementary proof can be given for example by modelling the proof of Theorem 1.9 in [13] in index 0 case. In what follows we outline the proof of the positive index case.

Proof. (Outline) The main difference here with the proof of [13, Theorem 1.9], is that we deal with closed loops, and that we can use in the context of surfaces a stronger form of the “automatic transversality” [13, Theorem 1.20]. (It becomes more automatic.) The setup with closed loops here has already been done in the author’s [12], but without the generality of Ustilovsky geodesics, and without the rather helpful “index theorem” 1.8.

Given our local unstable manifold (f_γ, E) , we obtain a Hamiltonian fibration

$$M \hookrightarrow P_{f_\gamma} \rightarrow B^k \times S^2,$$

by doing the standard clutching construction

$$M \times D^2 \sqcup_{\gamma_b} M \times D^2,$$

for a loop of diffeomorphisms of M , $\gamma_b = f_\gamma(b)$, as in the proof of Theorem 1.5, parametrically as b varies in B^k . For each b we obtain a natural coupling form $\tilde{\Omega}_b$ on $X_b \equiv P_{f_\gamma}|_{\{b\} \times S^2}$, as in the proof of Theorem 1.5. From construction it is immediate that the family $\{\tilde{\Omega}_b\}$ extends to a closed form $\tilde{\Omega}$ on P_{f_γ} and that this is a coupling form. Let $\{J_b\}$ denote the family of almost complex structures induced by the family $\{\tilde{\Omega}_b\}$ as in the proof of Theorem 1.5. The fixed point x_{\max} of γ gives a tautological flat and hence holomorphic section σ_{\max} of $X_0 \rightarrow S^2$, let us call by A its homology class in P_{f_γ} .

Let $\overline{\mathcal{M}}(P_{f_\gamma}, A)$ denote the compactified moduli space of pairs (u, b) , for u a J_b -holomorphic section of X_b in class A . Then by the same argument that is used in [13, Proposition 1.19], we get that $\overline{\mathcal{M}}(P_{f_\gamma}, A)$ consists of a single point $(\sigma_{\max}, 0)$. Let us briefly give an idea for this. For a given parameter b , by a kind of classical energy positivity, a J_b holomorphic section of X_b in class A will give a lower bound for the positive Hofer length of the loop $f_\gamma(b)$ and this lower bound is exactly $L^+(\gamma)$. So all the elements of our moduli space must appear in X_0 . Then a neat but simple trick, also based on energy positivity and originally due to Paul Seidel, allows one to conclude that σ_{\max} is the only possibility for such an element.

Let

$$D_{\sigma_{\max}}^{\text{rest}} : \Omega^0(\sigma_{\max}^* T^{\text{vert}} X_0) \rightarrow \Omega^{0,1}(\sigma_{\max}^* T^{\text{vert}} X_0),$$

be the associated real linear Cauchy-Riemann operator. Then we claim that this operator has trivial kernel. This works as follows. Twice of the vertical Chern number of the normal bundle $\sigma_{\max}^* T^{\text{vert}} X_0$ is less than -2 , as by construction this is the winding (Maslov) number of the linearization of γ at x_{\max} , and by assumption that the Morse index of γ is positive and by the “index” theorem 1.8, this winding number is less than -2 . Given this by the (proof of) the first part of [13, Theorem 1.20] and by the observation on the vanishing of the kernel of the CR operator above, $\overline{\mathcal{M}}(P_{f_\gamma}, A)$ can be regularized in such a way that it still consists only of $\{\sigma_{\max}\}$. In particular the associated parametric Gromov-Witten invariant is ± 1 , with the sign depending on how one chooses to orient the local unstable manifold.

Remark 3.3. *Parametric here is just emphasizing that we are not counting holomorphic curves in a symplectic manifold, but in a total space of a family of symplectic manifolds, but it is still a count in the usual sense of counting (with signs) zeros of a section of an associated Banach bundle. We refer the reader to [11] for further elaboration.*

The theorem follows, as we cannot deform the cycle f_γ to a cycle f'_γ lying in any lower sublevel set as that would preclude any A class elements (u, b) from existing in $\overline{\mathcal{M}}(P_{f'_\gamma}, A)$, since as we indicated above such an element gives a lower bound of exactly $L^+(\gamma)$ on $L^+(f'_\gamma(b))$ which would be absurd. \square

Proof of Theorem 1.10. Let's first do the case of $\text{Ham}(S^2, \omega)$. Suppose not, then we first observe that γ must be contractible as the minimal L^+ length of a non-contractible geodesic is $\hbar/2$. This is well known but can itself be deduced from Theorem 3.2, as the Hamiltonian S^1 action on S^2 representing generator of $\pi_1(\text{Ham}(S^2, \omega))$ satisfies the hypothesis, has positive Hofer length $\hbar/2$, and must be minimizing by the theorem. Next suppose that γ is a contractible quasi-integrable Ustilovsky geodesic and that the Morse index of γ is 0, in this case by Theorem 3.2 γ is minimizing, which is absurd unless γ is constant, so γ has positive Morse index k . Then again by Theorem 3.2:

$$\pi_k(\Omega^{\hbar/2} \text{Ham}(S^2, \omega), \Omega \text{Ham}(S^2, \omega)_E) \neq 0,$$

for some $E < L^+(\gamma) < \hbar/2$. Then using Theorem 1.5, we get that the inclusion map of $\Omega \text{Ham}(S^2, \omega)_E$ into $\Omega^{\hbar/2} \text{Ham}(S^2, \omega)$ is contractible from which it follows that

$$\pi_k(\Omega^{\hbar/2} \text{Ham}(S^2, \omega)) \neq 0,$$

but this contradicts Theorem 1.5.

For the case of $M = \Sigma$, if the index of a quasi-integrable Ustilovsky geodesic γ is 0 then again by Theorem 3.2 γ is minimizing which can only happen if γ is constant, as $\pi_1(\text{Ham}(\Sigma, \omega)) = 0$.

For a general L^+ Ustilovsky geodesic γ with positive Morse index k we get that

$$\pi_k(\Omega \text{Ham}(\Sigma, \omega), \Omega \text{Ham}(\Sigma, \omega)_E) \neq 0,$$

for some $E < L^+(\gamma)$, then using Theorem 1.5, we get that

$$\pi_k(\Omega \text{Ham}(\Sigma, \omega)) \neq 0,$$

but this is impossible since $\text{Ham}(\Sigma, \omega)$ is contractible. \square

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